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HEAT AND MASS TRANSFER LAWS FOR FULLY TURBULENT WALL FLOWS

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Abstract— The general method of Izakson and Millikan for the derivation of the well-known Prandtl–Nikuradse skin friction law is applied to the analysis of turbulent heat and mass transfer in pipes, channels, and boundary layers. The formula for the heat (or mass) transfer coefficient (or the Nusselt number) is obtained which contains the dimensionless coefficients of the universal logarithmic equations for the velocity and temperature profiles as parameters of the formula. One of these parameters is a universal function of Prandtl (or Schmidt) number and all the others are constants. The existing velocity and temperature profile measurements in various turbulent wall flows permit the determination of all the necessary coefficients with fair accuracy. The resulting calculations are in satisfactory agreement with numerous experiments on heat and mass transfer in pipes and boundary layers on a flat plate over the Prandtl (or Schmidt) number range from 6×10^{-3} to 10^6 and over two orders of magnitude of Reynolds (or Péclét) number variations.

NOMENCLATURE

a,	constant in the equation for the boundary layer thickness L;
a_1,	dimensionless thickness of a molecular diffusivity sublayer in the two-layer Prandtl–Taylor model;
a_M, a_H,	constants in the equations for ε_M and ε_H in the neighbourhood of a wall;
A,	$1/k$, constant in the universal velocity profile equation;
b_1, b_2,	constants in equations (26) and (27);
B, B_1,	constants in the universal velocity profile equations;
B_2, B_3, B_5,	constants in the universal skin friction laws;
B_4,	constant in equation (9);
c_f, C_f,	skin friction coefficients based on maximum velocity and bulk velocity, respectively;
c_h, C_h,	dimensionless heat or mass transfer, coefficients (i.e. thermal or diffusion Stanton numbers) based on maximum and bulk values, respectively;
c_p,	specific heat capacity at constant pressure (to be replaced by unity in case of mass transfer);
D,	$2L$, pipe diameter;
f, f_1, f_2,	universal functions;
j_w,	heat or mass flux at wall;
k,	$1/A$, Kármán constant;
L,	typical vertical size of the flow (channel halfwidth, pipe radius, or boundary layer thickness);
m,	exponent in the equations for ε_M and ε_H in the neighbourhood of a wall;
Nu, Nu_1,	$C_h Re Pr$, $c_h Re_1 Pr$, Nusselt numbers;
Pe, Pe_1,	$RePr$, $Re_1 Pr$, Péclét numbers;
Pr,	ν/χ, thermal or diffusion Prandtl numbers;
Pr_t,	$\varepsilon_M/\varepsilon_H$, turbulent Prandtl number inside the logarithmic layer;

Re, Re_1, Re_x, U_bL/ν or U_bD/ν, U_1L/ν, U_1x/ν; Reynolds numbers;
U, mean velocity;
u', v', components of the velocity fluctuation;
u_*, $(\tau_w/\rho)^{\frac{1}{2}}$, friction velocity;
x, coordinate measured normal to a of mean velocity; distance from leading edge of a plate;
y, coordinate measured normal to a wall;
y_+, yu_*/ν, dimensionless distance from a wall for wall region of the flow;
y_{1+}, y_{2+}, dimensionless thickness of the molecular diffusion sublayer and the viscous sublayer, respectively;
z, coordinate measured in direction normal to mean velocity;
α, APr_t, constant in the universal temperature (or concentration) profile equation;
$\beta(Pr)$, universal function of Pr entering into the temperature (or concentration) profile equation;
β_1, constant in the universal temperature (or concentration) profile equation;
β_2, β_3, constants in equations (19) and (20);
$\gamma(Pr), \gamma_1(Pr), \gamma_2(Pr)$, universal functions of Pr entering the heat and mass transfer laws;
Δ, correction factor defined by equation (32);
ε_M, eddy viscosity;
ε_H, eddy (thermal or mass) diffusivity;
ζ, u_*L/ν, Reynolds number based on friction velocity;
η, y/L, dimensionless distance from a wall for outer region of a flow;
η_1, function of Pe and C_f in equation (32);
Θ, mean temperature or concentration;
Θ_*, $j_w/c_p\rho u_*$, friction temperature (i.e. heat flux temperature or mass flux concentration);
Θ_+, $(\Theta_w - \Theta)/\Theta_*$, dimensionless temperature or concentration;
ν, kinematic viscosity;
ρ, density;
τ_w, wall shear stress;
$\varphi, \varphi_1, \varphi_2$, universal functions;
χ, molecular diffusivity for heat or mass transfer.

Subscripts

b, bulk quantities;
1, maximum quantities;
w, wall quantities;
$+$, dimensionless quantities.

1. INTRODUCTION

THIS paper is devoted to the derivation of a general heat and mass transfer law for a wide variety of turbulent flows along a flat smooth wall at sufficiently high Reynolds and Péclét numbers.* It will be assumed that the wall coincides with the plane $y = 0$, that the wall temperature (or wall concentration of a diffusing substance) is constant, and that the flow is a steady, parallel, turbulent flow in the x-axis direction with zero mean pressure gradient. Thus statistical homogeneity is assumed in the streamwise (x) as well as in the spanwise (z) direction. It is known that the stated assumptions are approximately valid with a reasonable accuracy for pipe and plane channel flow and for boundary layer flow along a flat plate without strong longitudinal pressure gradient. In the following, only these last three types of flow will be considered. It will be confined to the case of an incompressible fluid and a regime of dynamically passive (i.e. having no dynamic effect) heat or mass transfer. It then follows that in the heat transfer problems presented below, only forced convection conditions will be

* The preliminary announcement of the results of this study was published as a short note [1]. After the work was completed, it became known to the authors that a similar approach to the derivation of a heat and mass transfer law was also indicated (without any analysis of the data) in an unpublished lecture of Fortier [2].

considered, fluid properties will be taken as constant, and viscous dissipation will be neglected. Hence only the case of relatively low heat transfer density is relevant to the analysis. It is also clear that the assumption of streamwise homogeneity is satisfied only for a fully developed heat or mass transfer region; hence the thermal or diffusion entrance region of a flow must be excluded from consideration.

The principal physical processes in a turbulent flow along a horizontal wall are the vertical transfers of momentum, heat and mass. The momentum transfer is due to friction between fluid and wall; it is described by the dimensionless skin friction coefficient $c_f = 2(u_*/U_1)^2$ or $C_f = 2(u_*/U_b)^2$ where $u_* = (\tau_w/\rho)^{\frac{1}{2}}$ is the friction velocity, τ_w—wall shear stress, U_1—maximum mean velocity on a pipe or channel axis or a free stream velocity outside the outer edge of a boundary layer, and U_b—bulk velocity, i.e. the mean velocity averaged over the pipe or channel cross-section. The functional dependence of c_f or C_f on Reynolds number $Re_1 = U_1L/v$ or $Re = U_bL/v$ is called *the skin friction law* or *the resistance law*. The length scale L is a typical vertical size of the flow (for example, the channel halfwidth, the tube radius, or the boundary layer thickness).

Many different empirical skin friction laws have been proposed by various authors. However, the most interesting laws, from a physical point of view, are the Prandtl–Nikuradse skin friction law for pipe and channels flows and the closely related Kármán skin friction law for a boundary layer flow. Both of these laws can be justified theoretically with the aid of general similarity arguments. Such a derivation of the laws was proposed in the important papers [3, 4] which are based on the general idea put forward by Izakson [5]. This derivation can be found in a number of books and review papers (e.g. [6–8]). Since the derivation is the starting point of all the reasoning below, it seems reasonable to repeat it here briefly.

The similarity and dimensional arguments show that a mean velocity profile within a thin layer adjacent to the wall (at $y \ll L$) must satisfy the Prandtl wall law:

$$U(y) = u_* f\left(\frac{u_* y}{v}\right). \quad (1)$$

On the other hand, if Re_1 is high enough, then the Kármán velocity defect law must be valid in an outer turbulent region (at $y_+ = yu_*/v \gg 1$)

$$U_1 - U(y) = u_* f_1\left(\frac{y}{L}\right). \quad (2)$$

It is also clear that

$$U_1 = u_* f_2\left(\frac{u_* L}{v}\right). \quad (3)$$

Moreover an overlap layer exists at high enough Re_1 where both the laws (1) and (2) are valid simultaneously. If such a layer exists, then equations (1)–(3) imply the functional equation

$$f\left(\frac{u_* y}{v}\right) + f_1\left(\frac{y}{L}\right) = f_2\left(\frac{u_* L}{v}\right). \quad (4)$$

This equation was first obtained by Izakson [5] and it can be solved quite easily with the aid of successive differentiation with respect to y and L. The general solution of (4) has the form

$$f(y_+) = A \ln y_+ + B,\ f_1(\eta) = -A \ln \eta + B_1,$$
$$f_2(\zeta) = A \ln \zeta + B + B_1. \quad (5)$$

where $y_+ = yu_*/v$, $\eta = y/L$, and $\zeta = u_*L/v$. The experimental data show that $A = 1/k \approx 2{\cdot}5$ (i.e. Kármán constant $k \approx 0{\cdot}4$), $B \approx 5$ (but B is known less precisely than A), and B_1 is rather close to zero for pipe and channel flows while $B_1 \approx 2{\cdot}35$ for boundary layer flow along a flat plate (see e.g. [8]).

The last of equations (5) can be rewritten in the form

$$\sqrt{(2/c_f)} = A \ln (Re_1\sqrt{c_f}) + B_2,$$

$$B_2 = B + B_1 - A\frac{\ln 2}{2}. \quad (6)$$

Let us now note that the distance x from the leading edge of a plate is much easier to measure in the case of a boundary layer over a plate than the boundary layer thickness L. It is known that the relation $L = a(u_*/U_1)x$, where a = constant, is valid with satisfactory accuracy for zero pressure gradient boundary layer flow over a flat plate [8, 9]. Hence (6) implies the approximate equation

$$\sqrt{(2/c_f)} = A \ln (Re_x c_f) + B_3, \; Re_x = \frac{U_1 x}{\nu},$$

$$B_3 = B_2 + A \ln \frac{a}{\sqrt{2}}. \qquad (7)$$

This is the Kármán skin friction law for boundary layer flows. It agrees well with experiments on setting $B_3 \approx 2{\cdot}4$.

In the case of pipe and channels flows almost all the data are given with reference to the variables C_f and Re and not c_f and Re_1. Hence the friction law (6) must be transformed accordingly using the equations

$$U_b = \frac{1}{L}\int_0^L U(y)\,\mathrm{d}y$$

and (8)

$$U_b = \frac{2}{L^2}\int_0^L (L - y)\,U(y)\,\mathrm{d}y$$

the first of which refers to channel flow and the second to pipe flow. Let us neglect the thin wall layer of the direct molecular viscosity influence (i.e. viscous and buffer sublayers). In other words, let us suppose that the velocity defect law (2) is valid at any y: this supposition is quite justified if Re_1 is sufficiently high. Then equations (2) and (8) yield the relation

$$\frac{U_1 - U_b}{u_*} = B_4 = \text{constant}, \qquad (9)$$

where $B_4 = \int_0^1 f_1(\eta)\,\mathrm{d}\eta$ for channel flow and $B_4 = 2\int_0^1 (1-\eta) f_1(\eta)\,\mathrm{d}\eta$ for pipe flow. If we assume that the logarithmic form of a velocity defect law is valid up to the pipe or channel axis (and hence $B_1 = 0$), then $B_4 = A \approx 2{\cdot}5$ for channel flow and $B_4 = 1{\cdot}5\,A \approx 3{\cdot}75$ for pipe flow. These estimates of B_4 have fair precision, since the assumption introduced is sufficiently accurate for fully turbulent pipe and channel flows. Equations (6) and (9) imply the relation

$$\sqrt{(2/C_f)} = A \ln (Re \sqrt{C_f}) + B_5 \qquad (10)$$

where $B_5 = B_2 + B_4$. Equation (10) is the famous Prandtl–Nikuradse skin friction law for pipe and channel flow.

Let us now consider turbulent heat or mass transfer from the wall. The main dimensionless characteristic of such a transfer is the heat or mass transfer coefficient (identical to the thermal or diffusion Stanton number)

$$c_h = \frac{j_w}{c_p \rho U_1 (\Theta_w - \Theta_1)}$$

or

$$C_h = \frac{j_w}{c_p \rho U_b (\Theta_w - \Theta_b)}$$

where j_w is the heat or mass flux at the wall, Θ_w is the wall temperature or wall concentration of the diffusing species, Θ_1—temperature or concentration at the axis or at the outer edge of a boundary layer, Θ_b—bulk temperature or concentration, and c_p—specific heat capacity at constant pressure (it is supposed here and in all subsequent equations that $c_p = 1$ for the mass transfer problem). The Nusselt number $Nu_1 = c_h Re_1 Pr$ or $Nu = C_h Re Pr$ is often used instead of c_h or C_h: here $Pr = \nu/\chi$ is the thermal or diffusion Prandtl number (i.e. Prandtl or Schmidt number). The dependence of c_h or Re_1 and Pr or C_h on Re and Pr is called *the heat or mass transfer law*. Many empirical forms of the heat and mass transfer law can be found in the literature. They are valid for different ranges of Reynolds and Prandtl numbers and for various types of turbulent flows. It will be shown below that the general theoretically justified form of the laws can be established very simply quite similarly to the above derivation of Kármán and Prandtl–Nikuradse skin friction laws. The obtained equation contains some unknown quantities, but all of them can be determined at present with fair accuracy from existing experimental data.

2. DERIVATION OF THE LAWS

The natural temperature or concentration scale $\Theta_* = j_w/c_p\rho u_*$ was introduced (for temperature only) by H. Squire [10] in 1951 who called it a friction temperature *. We think however that the term heat flux temperature (in the heat transfer case) or mass flux concentration (in the mass transfer case) would be more appropriate. In a wall layer (at $y \ll L$) the vertical length scale L cannot play a role at all; hence the temperature or concentration distribution within the wall layer is determined by the quantities j_w, ρ, c_p, u_*, ν and χ. The dimensional arguments yield the thermal or diffusion wall law valid for a wall layer

$$\Theta_w - \Theta(y) = \Theta_* \varphi\left(\frac{u_* y}{\nu}, Pr\right). \tag{11}$$

This law was first formulated by H. Squire [10]. On the other hand, if the Reynolds number Re_1 and Péclét number $Pe_1 = Re_1 Pr$ are high enough then the turbulent heat or mass transfer will be much greater than the molecular transfer in an outer turbulent region of a flow. Consequently the temperature or concentration differences in the outer region must be independent on the molecular constants ν and χ. This is a general Reynolds and Péclét number similarity principle which is quite analogous to the well known Reynolds number similarity principle. It follows that the temperature or concentration defect law must be valid in the outer turbulent region

$$\Theta(y) - \Theta_1 = \Theta_* \varphi_1\left(\frac{y}{L}\right). \tag{12}$$

* The same quantity was in fact also used without any special name by Landau and Lifshitz in 1944 (in the first Russian edition of the book [9]) and by Obukhov [11] in 1946.

W. Squire [12] was apparently the first to suggest this law. Both the laws (11) and (12) are less widely known than the analogous velocity laws (1) and (2). However they have the same degree of theoretical confidence and are quite reliably confirmed by experiments (see section 3 below).

It is clear that the total difference of the field $\Theta(y)$ between a wall and flow axis (or outer edge of a boundary layer) must be a function of u_*L/ν and $Pr = \nu/\chi$:

$$\Theta_w - \Theta_1 = \Theta_* \varphi_2\left(\frac{u_* L}{\nu}, Pr\right). \tag{13}$$

If Reynolds and Péclét numbers are both high enough, then an overlap layer must exist when both the laws (11) and (12) are valid. Within this layer equations (11)–(13) imply a functional equation of a form

$$\varphi\left(\frac{u_* y}{\nu}, Pr\right) + \varphi_1\left(\frac{y}{L}\right) = \varphi_2\left(\frac{u_* L}{\nu}, Pr\right). \tag{14}$$

This equation coincides with equation (4) only if the dependence on the first argument is taken into account. Hence its general solution is of the form:

$$\varphi(y_+, Pr) = \alpha \ln y_+ + \beta(Pr),$$
$$\varphi_1(\eta) = -\alpha \ln \eta + \beta_1,$$
$$\varphi_2(\zeta, Pr) = \alpha \ln \zeta + \beta(Pr) + \beta_1. \tag{15}$$

The first of equations (15) is a logarithmic equation for a mean temperature or concentration profile near a wall which was obtained for the first time by Landau and Lifshitz in 1944 (in the first Russian edition of the book [9]). The second equation describes a logarithmic form of the defect law. Finally the third equation is a heat or mass transfer law for a turbulent wall flow. It can be rewritten in the form

$$c_h = \frac{\sqrt{(c_f/2)}}{\alpha \ln (Re_1 \sqrt{c_f}) + \gamma(Pr)} \tag{16}$$

where $\gamma(Pr) = \beta(Pr) + \beta_1 - (\alpha \ln 2)/2$. The friction coefficient c_f is a known function of Re_1 by virtue of (6), and hence equation (16) determines the form of the dependence of c_h on Re_1 and Pr. For the use in calculations of (16), we must also know the constant α and the function $\gamma(Pr)$. The problem of the determination of these quantities will be discussed in detail in the next section.

For many engineering applications and for the comparison with experimental results, some alternative forms of equation (16) are useful. Let us first consider the calculation of the local heat and mass transfer in the fully developed turbulent boundary layer on a flat plate where the temperature or concentration boundary layer thickness is practically the same as the thickness of the velocity boundary layer. Let us replace the boundary layer thickness L by the more easily measured distance x from the leading edge of the plate in this case. Since $x = a^{-1}\ U_1 L/u_*$, equation (16) is transformed after such a substitution into the form

$$c_h = \frac{\sqrt{c_f/2}}{\alpha \ln (Re_x c_f) + \gamma_1(Pr)} \qquad (17)$$

where $Re_x = U_1 x/\nu$, $\gamma_1(Pr) = \gamma(Pr) + \alpha(B_3 - B_2)/A$, and B_3, B_2 and A are constants in equations (6) and (7). At any fixed value of Pr the heat or mass transfer law (17) contains two unknown numerical coefficients, α and γ_1: it is quite analogous to the Kármán skin friction law (7). The term $\alpha \ln(Re_x c_f)$ in the denominator of the right-hand side of (17) can be replaced by $(\sqrt{2/c_f} - B_3)\alpha/A$ by virtue of (7).

In the case of pipe and channel flows the heat and mass transfer data are always with reference to the variables C_h and C_f and not c_h and c_f. In other words, the bulk velocity U_b and bulk temperature or concentration Θ_b are used here instead of axial quantities U_1 and Θ_1. The replacement of U_1 by U_b produces no change in the form of equation (16) since it is easily seen that the velocity scale can be chosen quite arbitrary in this equation. Hence we can assume that the coefficient c_h in the left-hand side of the equation (16) is determined by the equation $c_h = j_w/c_p \rho U_b(\Theta_w - \Theta_1)$ and the coefficient c_f in the right-hand side replaced by $C_f = 2(u_*/U_b)^2$. However the transformation from Θ_1 to Θ_b is more complicated. Let us use one of the equations

$$\Theta_b = \frac{1}{U_b L}\int_0^L \Theta(y)U(y)\mathrm{d}y$$

and (18)

$$\Theta_b = \frac{2}{U_b L^2}\int_0^L (L - y)\Theta(y)U(y)\mathrm{d}y$$

where the first of them applies to channel flow, and the second to pipe flow. Let us also assume that the numbers Re and $Pe = RePr$ are so high that the wall sublayer where the molecular transfers are important is very thin and that it does not contribute significantly to the formation of the mean profiles $\Theta(y)$ and $U(y)$. Then it is possible to neglect this sublayer, i.e. to assume that defect laws (2) and (12) are valid at any y. Let us now replace the functions $\Theta(y)$ and $U(y)$ in equations (18) by their expressions implied by (12) and (2) and use equation (9). Then we easily obtain that

$$\frac{\Theta_b - \Theta_1}{\Theta_*} = \beta_2 - \beta_3\sqrt{(C_f/2)} \qquad (19)$$

and consequently

$$\frac{\Theta_w - \Theta_b}{\Theta_w - \Theta_1} = 1 - \frac{\Theta_*}{\Theta_w - \Theta_1}\frac{\Theta_b - \Theta_1}{\Theta_*}$$
$$= 1 - \frac{c_h}{\sqrt{(C_f/2)}}[\beta_2 - \beta_3\sqrt{(C_f/2)}] \qquad (20)$$

where

$$\beta_2 = \begin{cases} \int_0^1 \varphi_1(\eta)\mathrm{d}\eta & \text{for channel flow,} \\ 2\int_0^1 (1-\eta)\varphi_1(\eta)\mathrm{d}\eta & \text{for pipe flow,} \end{cases} \qquad (21a)$$

and

$$\beta_3 = \begin{cases} \int_0^1 \varphi_1(\eta) f_1(\eta)\mathrm{d}\eta - B_4\beta_2 & \text{for channel flow.} \\ 2\int_0^1 (1-\eta)\varphi_1(\eta) f_1(\eta)\mathrm{d}\eta - B_4\beta_2 & \text{for pipe flow.} \end{cases} \qquad (21b)$$

If we accept that the logarithmic form of the defect laws for both velocity and temperature or concentration are approximately valid up to the axis (so that $B_1 = \beta_1 = 0$), then all the integrals in the right-hand sides of (21) can be integrated explicitly and we obtain

$$\beta_2 = \alpha, \quad \beta_3 = A\alpha \quad \text{for channel flow.}$$
$$\beta_2 = 1{\cdot}5\alpha, \beta_3 = 1{\cdot}25A\alpha \quad \text{for pipe flow.} \tag{22}$$

The assumption used for deriving the estimates (22) agree satisfactorily with all the existing data and hence these estimates are apparently rather accurate. Let us also note that fully developed turbulent flow with an appreciable overlap layer can apparently exist only when Re is not less than 4×10^3 and the corresponding value of $(C_f/2)^{1/2}$ will be then of the order of 0·07 or less. Hence the second term in the brackets on the right-hand side of (20) may be neglected in comparison with the first term. If this term is preserved, then it leads to slight modifications in all the subsequent equations which are however of no importance for real applications and will not therefore be considered in this paper.

Now we can use the equation

$$C_h = \frac{j_w}{c_p \rho U_b(\Theta_w - \Theta_1)} \frac{\Theta_w - \Theta_1}{\Theta_w - \Theta_b}$$

where the first factor in the right-hand side is given by equation (16) with c_f replaced by C_f and the second factor is given by the equation (20). If we neglect additionally the term $\beta_3(C_f/2)^{1/2}$ in (20) then we finally obtain

$$C_h = \frac{\sqrt{(C_f/2)}}{\alpha \ln (Re\sqrt{C_f}) + \gamma_2(Pr)} \tag{23}$$

where $\gamma_2(Pr) = \gamma(Pr) - \beta_2$. If the Reynolds number of a pipe flow is defined as $Re = U_b D/\nu = 2U_b L/\nu$ (instead of $U_b L/\nu$) then the function $\gamma_2(Pr)$ contains a supplementary term $-\alpha \ln 2$; this convention for Re will be used in the next section throughout. The heat and mass transfer law (23) contains two unknown constants α and γ_2 at any fixed value of Pr. It is quite analogous to the classical Prandtl–Nikuradse skin friction law (10). The term $\alpha \ln (Re\sqrt{C_f})$ in the denominator of (23) can be replaced by $[\sqrt{(2/C_f)} - B_5]\alpha/A$ by virtue of (10).

Some particular heat and mass transfer laws were previously suggested which are in fact special cases of the general equations (16) and (23) (see e.g. [13–15]). These equations were obtained with the aid of approximate semi-empirical calculations suitable for a restricted range of Pr values or were selected as empirical interpolation formulae; moreover, the particular forms of the functions $\gamma(Pr)$ and $\gamma_2(Pr)$ recommended in all these publications disagree strongly with recent experiments. We also hold that the preceding general derivation of the laws is worth attention since it is based on the firm physical basis of dimensional analysis and shows that equation (16) is a direct consequence of the usual assumption about the validity of wall and defect laws and a single additional assumption of the existence of an overlap layer. The derivation of equation (23) requires additionally that the contribution of the viscous and molecular diffusivity sublayers and of the buffer sublayer to the bulk velocity and bulk temperature or concentration is negligible. Without this assumption the equation cannot be justified. The stated assumption (and therefore equation (23)) is clearly inadmissible in all practicable problems with $Pr \ll 1$, i.e. in the liquid metal range of heat transfer problems. It will be shown below that the general equation (16) can be used for the derivation of the liquid metal heat transfer law; however, the law is in this case more complicated than (23). It is also important to note that the coefficients $\gamma(Pr)$, $\gamma_1(Pr)$ and $\gamma_2(Pr)$ in equations (16), (17) and (23) have a clear physical meaning and can be determined from the profile measurements independently from heat and mass transfer problems. The possibility of independently determining all of the coefficients permits sufficiently reliable verification of the equations with existing data. We now pass on to this verification of the theoretical equations.

3. COMPARISON WITH EXPERIMENTS

Let us now consider the problem of the determination of the universal constants and functions of Pr in the heat and mass transfer laws from experimental data and the comparison of the equations with experimental results.

3.1 *Heat transfer at* Pr = 0·7 (*air*)

We shall begin with the determination of the constant α. It is easily seen that $\alpha = APr_t = Pr_t/k$ where $k = 1/A \approx 0{\cdot}4$ is the Kármán constant and $Pr_t = \varepsilon_M/\varepsilon_H$ is the turbulent Prandtl number within the logarithmic layer. Hence the determination of α is closely related to the determination of $Pr_t = k\alpha$. There are many discrepancies in the determinations of Pr_t which can be found in the literature. The data collected in the review paper by Kestin and Richardson [16] force one to think that apparently Pr_t is within or close to the range 0·9–0·7, but the dependence of Pr_t on the distance from the wall turns out to be quite variable according to the data of different authors. The data on Pr_t collected in a more recent review of Blom and De Vries [17] do in fact fill all the range from 0·5 up to 1·5 as a continuous cloud, yield numerous contradictory (and even fantastic) forms of a dependence of Pr_t on y and show substantial dependence of Pr_t on $Pr = \nu/\chi$. It is worth stressing once again in this respect that the arguments of section 2 show that Pr_t must be constant in the logarithmic layer, i.e. within this layer it cannot be dependent on y at all. Moreover, if one accepts the Reynolds and Péclét number similarity then it follows that Pr_t cannot be dependent on Pr, and if this similarity is not accepted, then it is impossible to explain satisfactorily the validity of the logarithmic profile equations. Finally, if we accept that heat can be considered as a dynamically passive substance, then the value of Pr_t for heat and for mass must be the same.

We are inclined to think that the very great scatter of the data on Pr_t is mainly connected with the necessity of differentiating two measured profiles and determining the shear stress and heat flux profiles for calculation of the turbulent Prandtl number. All these procedures clearly entail great errors. Moreover, the determination of the number Pr_t should be made inside the logarithmic region of the flow which is usually physically quite thin. Thus outside points are often used. Therefore we prefer to use another approach. There are many works at present which show quite reliably the existence of a layer having a logarithmic temperature profile. There are also works where the logarithmic profile was not suggested by the authors, but it can be obtained quite accurately if the given data are analysed. The last situation is, for example, quite usual for many investigations of liquid metal heat transfer (see, e.g. Fig. 1). Values of coefficients α and β are easily determined from a logarithmic temperature profile if the values of the quantities j_w and u_* and the fluid properties are also known. All such data collected by us are represented in Table 1. The values given here of Pr_t were calculated with the value of k given in the indicated paper or with the assumption $k = 0{\cdot}4$ if no value is given.

The spread of the values of α and Pr_t in Table 1 is evidently not greater than the spread in the values of the Kármán constant k obtained by different experimentalists. The values of α and Pr_t turn out to be practically independent of Pr as expected. The most reliable of the collected data lead to the conclusion that $Pr_t \approx 0{\cdot}85$, $\alpha \approx 2{\cdot}12$. These values of Pr_t and α will be used in all the following considerations.

The second necessary constant β is a universal function of the Prandtl number Pr. The data of Table 1 for heat transfer through air (at practically constant value $Pr = 0{\cdot}7$) show that the experimental values of $\beta(0{\cdot}7)$ are scattered no more than values of the constant term B in the logarithmic velocity profile equation. They are the basis for setting $\beta(0{\cdot}7) \approx 3{\cdot}8$. The value of the constant term β_1 in the logarithmic temperature defect equation is independent of Pr, but can be different for flows in pipes, channels and boundary layers. For pipe flow β_1 is comparatively close to zero according to the data of

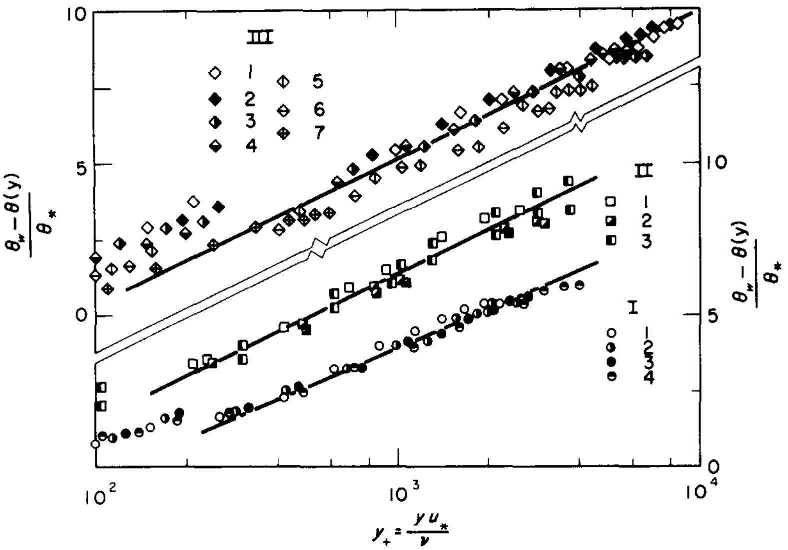

FIG. 1. Measured temperature profiles in turbulent mercury flow in a pipe.
I. Reference [18], $Pr = 0{\cdot}022$. 1. $Re = 89200$, 2. $Re = 101000$, 3. $Re = 114800$, 4. $Re = 177000$.
II. References [19, 20], $Pr = 0{\cdot}026$. 1. $Re = 107000$, 2. $Re = 130000$, 3. $Re = 165000$.
III. Reference [21], $Pr = 0{\cdot}026$. 1. $Re = 427000$, 2. $Re = 373000$, 3. $Re = 328000$, 4. $Re = 274000$, 5. $Re = 205000$, 6. $Re = 174500$, 7. $Re = 28000$.

Table 1. Experimental values of the coefficients in the logarithmic temperature equation

Authors and date	Ref.	Fluid	Pr	Re: Re_x $\times 10^{-3}$	α	Pr_t	β	Type of flow
1	2	3	4	5	6	7	8	9
Borishanskii *et al.* (1964)	[18]	mercury	0·022	89–177	2·00	0·80	− 9·2	pipe
Kokorev, Ryaposov (1962)	[19, 20]	mercury	0·026	107–167	2·09	0·84	− 7·9	pipe
Subbotin *et al.* (1963)	[21]	mercury	0·026	22–427	2·11	0·85	− 9·0	pipe
Seban, Shimazki (1951)	[22]	air	0·7	5–54	2·04	0·82	2·9	pipe
Deissler, Eian (1952)	[14]	air	0·7	—	2·18	0·87	3·8	pipe
Reynolds *et al.* (1958)	[23]	air	0·7	729–2780	2·00	0·80	4·0	boundary layer
Johnk, Hanratty (1962)	[24]	air	0·7	18–71	2·22	0·85	3·3	pipe
Brundrett *et al.* (1965)	[25]	air	0·7	34–76	1·96	0·79	3·8	square channel
Perry *et al.* (1966)	[23]	air	0·7	1800	2·00	0·80	4·0	boundary layer
Achenbach (1966)	[26]	air	0·7	300–2000	1·80	0·72	3·9	boundary layer
Hishida (1967)	[27]	air	0·7	10–81	2·17	0·87	3·8	pipe
Gowen, Smith (1967)	[14]	air	0·7	16–49	2·18	0·87	3·0	pipe
Taccoen (1968)	[28]	air	0·7	70–560	2·19	0·83	4·2	pipe
Pedisius *et al.* (1969)	[29, 30]	air	0·7	340–870	1·81	0·75	3·8	boundary layer
Pedisius *et al.* (1969)	[29, 30]	water	3·0	870–4400	1·95	0·80	26·0	boundary layer
		water	5·5	400–2400	2·02	0·83	41·4	
Smith *et al.* (1967)	[31, 14]	water	5·7	11–44	2·58	1	34·5	pipe
Beckwith *et al.* (1963)	[14]	water	6·0	11–19	2·55	1	28·0	pipe
Che Pen Chen (1969)	[32]	water	7·5	250–280	2·12	0·85	47	plane channel
Gowen, Smith (1967)	[14]	ethylene-glycol	14·3	16–25	2·52	1	76·3	pipe
Pedisius *et al.* (1969)	[29, 30]	technical oil	64	350–850	2·13	0·87	194	boundary layer
Neumann (1968)	[33]	technical oil	60	34·3			200	plane channel
			67	89·0			227	
			80	32·8	2·19	0·87	251	
			95	27·0			275	
			100	41·6			280	
			103	57·9			293	

[14, 18–21, 24] (the data of [18–21] and of [24] give the impression that apparently $\beta_1 \approx 0{\cdot}6$–$0{\cdot}8$ while in [14] the estimate $\beta_1 = -0{\cdot}4$ is used). In the present work we shall assume that $\beta_1 = 0{\cdot}6$, since any sufficiently small value of β_1 yields practically the same results for $Pr \geqslant 0{\cdot}5$, and the data at $Pr \ll 1$ which lead to much stronger dependence on β_1 give the impression that β_1 is small in absolute value but nevertheless strictly positive. Unfortunately no data are known to us which permit one to obtain even a rough estimate of the value of β_1 for a boundary layer along a flat plate. Therefore we confine ourselves to the approximate estimate implied by the widely used assumption that the eddy diffusivities for heat and for momentum are equal in the outer turbulent region of the flow (so-called Reynolds analogy). If the assumption is correct then the dimensionless velocity and temperature profiles have the same shape in the outer turbulent region and consequently β_1 for boundary layer flow must coincide with the constant term $B_1 \approx 2{\cdot}35$ in the logarithmic velocity defect law for a boundary layer along a plate.

If we take all above mentioned values of the coefficients and also use the approximate relation (22), we obtain the following form of the heat transfer law for turbulent pipe flow in air:

$$Nu = C_h \, Re \, Pr = \frac{Re\sqrt{(C_f)}}{4{\cdot}3 \ln [Re\sqrt{(C_f)}] - 2{\cdot}0}. \quad (24)$$

We recall that all the coefficients in this equation were determined from profile measurements without any reference to heat transfer data. The dependence of Nu on Re implied by equation (24) is shown as a continuous line in Fig. 2. The points in this figure represent the experimental data of the works listed in Table 2 and a dotted line gives the empirical law $Nu = 0{\cdot}018\ Re^{0{\cdot}8}$ which is implied by the approximate relations recommended in textbooks [15] and [58]. We

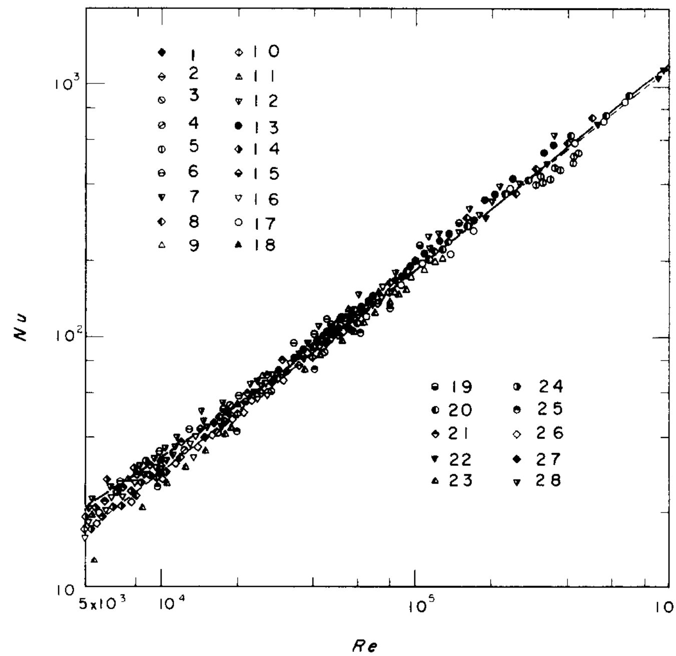

FIG. 2. Nu for air flow in pipes vs. Re.
All numbers correspond to relevant data in Table 2. The continuous line represents equation (24) and the dotted line gives the empirical law $Nu = 0{\cdot}018\ Re^{0{\cdot}8}$.

Table 2. Heat transfer studies for turbulent air flows in pipes

No.	Authors and date	Ref.	$Re \times 10^{-3}$
1	2	3	4
1.	Coglan (1940)	[34]	10
2.	Drexel, McAdams (1945)	[24, 34]	10–70
3.	Boelter *et al.* (1948)	[35]	27–55
4.	Cholette (1948)	[36]	5–18
5.	Gukhman *et al.* (1949)	[37, 38]	300–450
6.	Il'in (1951)	[39]	6–70
7.	Seban, Shimazki (1951)	[22]	5–54
8.	Pincel (1954)	[40]	8–500
9.	Seleznev (1956)	[41]	83–128
10.	Nunner (1956)	[42]	5–80
11.	Koch (1958)	[43]	5–80
12.	Sleicher (1958)	[44]	14–80
13.	Lel'chuk, Dyadyakin (1959)	[45]	33–350
14.	Mikheev (1959)	[46]	5–10
15.	Abbrecht, Churchill (1960)	[47]	15–65
16.	Ede (1961)	[48]	5–100
17.	Mukhin *et al.* (1962)	[49]	40–700
18.	Johnk, Hanratty (1962)	[24]	18–71
19.	Kirillov, Malgin (1963)	[50]	7–160
20.	Petukhov *et al.* (1963)	[51]	40–700
21.	Novozhilov *et al.* (1964)	[52]	6–38
22.	Delpont (1964)	[53]	180–970
23.	Kolař (1965)	[54]	5–92
24.	Gowen, Smith (1967)	[14]	10–49
25.	Hishida (1967)	[27]	10–81
26.	Dyban, Epik (1968)	[55]	5–50
27.	Hasegawa, Fujita (1968)	[56]	16–50
28.	Sukomel, Velichko (1969)	[57]	5–350

see that equation (24) deviates insignificantly from the empirical law in the range $2 \times 10^4 \leqslant Re \leqslant 10^6$ and the curve fits all the data quite well in this range. At $Re < 10^4$ the empirical law fits the data slightly better than the theoretical law (24). This can be reasonably explained by the fact that at such small Re (and consequently at $Pe = Re\,Pr < 7 \times 10^3$) the universal logarithmic law is not sufficiently accurate (cf. [59])*. At $Re > 5 \times 10^6$ equation (24) will differ significantly from the empirical law. Unfortunately there are no data at present concerning heat transfer in pipes at such values of Re.

Let us now pass on to the problem of heat transfer through an air boundary layer over a flat plate. The general equation (17) supplemented by the above recommendations concerning the values of the coefficients α, A, β, β_1, B_2, B_3 and Pr yield the following form of a heat transfer law:

$$Nu = c_h Re_x Pr = \frac{Re_x \sqrt{c_f}}{4{\cdot}3 \ln(Re_x c_f) + 3{\cdot}8}. \qquad (25)$$

Equation (25) is shown by a continuous line in Fig. 3, together with data from the work listed in

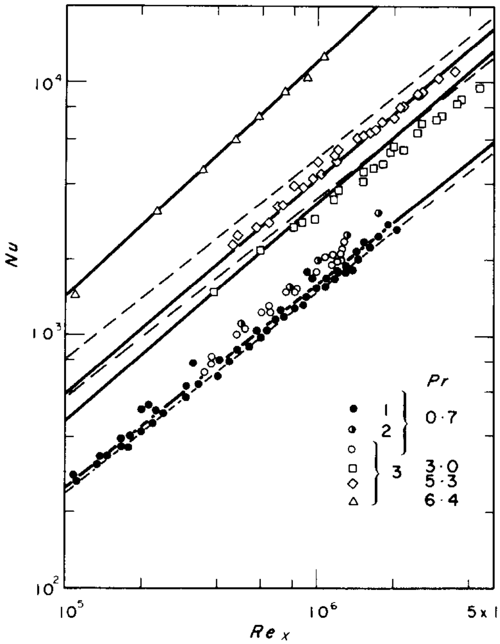

FIG. 3. Nu for air boundary layer over a flat plate vs. Re_x. All numbers correspond to relevant data in Table 3. The continuous lines represent equation (29) and the dotted lines represent the empirical equation suggested by Kays [15].

Table 3, and the empirical law $Nu = 0{\cdot}0238\, Re_x^{0{\cdot}8}$, deduced from the experimental data of Reynolds, Kays and Kline (see Kays [15]). We see that the data are rather scattered, but in the average equation (25) fits them quite satisfactorily.

* It is known that the universal skin friction law of Prandtl and Nikuradse fits all the data on friction in pipes quite satisfactorily beginning from a substantially low value of Reynolds number of the order of 5×10^3. However, it is also known that C_h reaches the asymptotic value for fully turbulent flow only at Re of the order of 2×10^4, i.e. considerably later than the asymptotic value of C_f is established (cf. e.g. [14]).

Table 3. Local heat transfer studies for turbulent boundary layers over a flat plate

No.	Authors and date	Ref.	Fluid	Pr	$Re_x \times 10^{-3}$
1.	Petukhov *et al.* (1954)	[60]	air	0·7	100–2000
2.	Achenbach (1966)	[26]	air	0·7	320–1700
3.	Slanciauskas *et al.* (1969)	[30]	air	0·7	350–1300
			water	3·0	400–4400
				5·3	460–3500
			technical oil	64	110–1050

We do not have data on the heat transfer through the air boundary layer over a flat plate at $Re_x > 2 \times 10^6$. However the data of the thorough measurements of Survila and Stasiulevicius [61] may be used. They studied heat transfer through an air turbulent boundary layer on a cylinder streamlined in the direction of its axis. This is completely justified since a special investigation of the above mentioned authors [62] has shown that the surface curvature does not effect the heat transfer in their experiments. The data obtained are shown in Fig. 4. We see that they are described rather accurately by equation (25), but the empirical equation of Kays does not fit them.

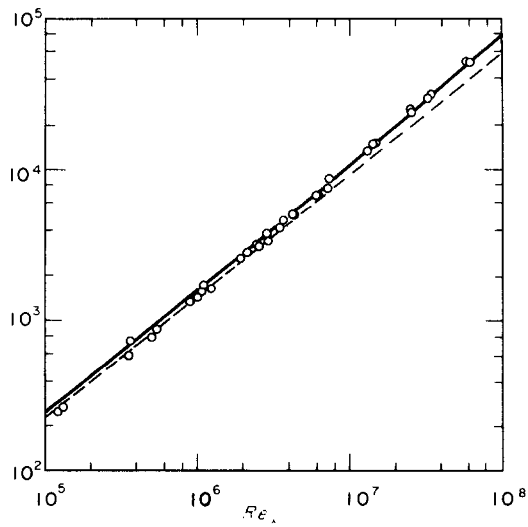

FIG. 4. Nu for a boundary layer on a cylinder streamlined in the direction of its axis vs. Re_x.
The continuous line represents equation (25) and the dotted line gives the empirical equation of Kays [15].

3.2 *Heat and mass transfer at* Pr > 0·5

In order to conduct the calculation at any value of Pr, we must know the function $\beta(Pr)$. This function enters several previously established methods of the heat and mass transfer calculations and this is the reason why there are several previous suggestions concerning its form. All these suggestions are listed in Table 4 together with references to the relevant papers. Equation (1) was suggested by H. Squire [10] as far back as 1951. He justified it by Kármán's well-known three-layer model of a turbulent boundary layer (the laminar sublayer without any turbulence at $0 < y_+ < 5$, buffer layer with an eddy viscosity $\varepsilon_M = 0{\cdot}5ku_*y$ at $5 < y_+ < 30$, and logarithmic layer with $\varepsilon_M = ku_*y$ at $y_+ > 30$). Squire followed Kármán by assuming that $\varepsilon_M/\varepsilon_H = 1$ at any y, but later Gowen and Smith [14] considered a slight modification of Squire's equation which is obtained if one assumes that $Pr_t \neq 1$ within a logarithmic layer (equation Ia). Equation II is an empirical equation suggested by Neumann [33] for describing his data on temperature profiles in a plane channel flow of technical oil. Prandtl number was varied in these experiments within a relatively narrow range $70 < Pr < 110$ as the result of variations in the mean temperature of the oil. Equation III is due to Fortier [2] (see also Taccoen [64]). This equation is based on an important result (established at the first time apparently by Levich [13]) that $\beta(Pr) = \alpha \ln Pr +$ constant at

Table 4. Proposed equations for $\beta(Pr)$

Authors and date	Ref.	Equation	Number
1	2	3	4
H. Squire (1951)	[10]	$\ln[(5Pr + 1)/30] + 8{\cdot}55 + 5Pr$	I
Gowen, Smith (1967)	[14]	$\ln[(5Pr + 1)/30] + 8{\cdot}55Pr_t + 5Pr$	Ia
Neumann (1968)	[33]	$3{\cdot}5Pr$	II
Fortier (1968)	[2]	$2{\cdot}5 \ln Pr + 2{\cdot}7\, Pr - 1$	III

$Pr \ll 1$. Fortier used the liquid metal data of Kirillov [65] which show that $\beta(Pr) \approx \alpha \ln Pr - 1$, $\alpha = A = 2{\cdot}5$ when $Pr \ll 1$, and supplemented this equation by an additional linear term chosen to fit Neumann's technical oil data.

The comparison of all of the equations from Table 4 with the experimental data of $\beta(Pr)$ collected in Table 1 is shown in Fig. 5 where equation Ia is used with the values of Pr_t from [14]. We see that neither of the suggested equations fits the data. Let us also note that all the equations imply that $\beta(Pr) \sim Pr$ at $Pr \gg 1$. Such behaviour of $\beta(Pr)$ corresponds to the assumption that at the wall there exists a laminar sublayer of a fixed dimensionless thickness y_+ where there is no turbulence at all (see [13, 64]). This assumption strongly contradicts all the recent data on viscous sublayer flow and will evidently yield incorrect results when applied to heat and mass transfer calculations in the range of very high Pr.

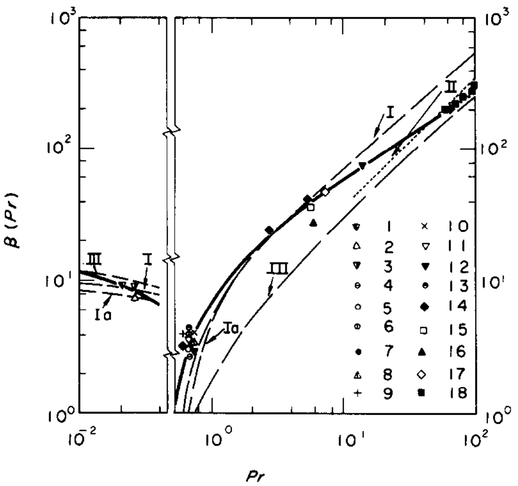

FIG. 5. Comparison of data on $\beta(Pr)$ with the suggested empricial equations. The solid lines correspond to equations (28) and (31). The Roman numbers are the same as in Table 4. The data are from the following works:

1.—ref. [18], 2.—ref. [19, 20], 3.—ref. [21], 4.—ref. [22], 5.—ref. [14], 6.—ref. [23], 7.—ref. [24], 8.—ref. [25], 9.—ref. [23], 10.—ref. [26], 11.—ref. [27], 12.—ref. [14], 13.—ref. [28], 14.—ref. [29, 30], 15.—ref. [31, 14], 16.—ref. [14], 17.—ref. [32], 18.—ref. [33].

There is no divergence of opinion at present that both the eddy viscosity ε_M and the eddy diffusivity ε_H are nowhere equal to zero, but decrease monotonically according to a power law $\varepsilon_M \sim \varepsilon_H \sim y^m$ as $y \to 0$. In other words, the relations $\varepsilon_M/\nu = a_M y_+^m$ and $\varepsilon_H/\nu = a_H y_+^m$ apply in the neighbourhood of the wall. There is no complete agreement concerning the value of the exponent m (see the review of the related investigations in subsection 5.7 of the book [8]). More recently, accurate experiments of several authors give strong support to the value $m = 3$ (see, in particular [66–72]). This last value of m will be used in the present work.

The general approach to the determination of the behaviour of $\beta(Pr)$ at sufficiently high values of Pr is sketched by Levich [13], who, however, does not use the function $\beta(Pr)$ in his book. The method is based on the use of a particular simple three-layer model of a boundary layer. Namely, it is assumed that adjacent to the wall is a thin "molecular diffusion sublayer" where the molecular transfers play a dominant role, and therefore the dimensionless profile $\Theta_+(y_+) = [\Theta_w - \Theta(y_+ \nu/u_*)]/\Theta_*$ is given by the equation $\Theta_+(y_+) = Pr y_+$. The dimensionless thickness y_{1+} of the sublayer can be determined from the condition that $\varepsilon_H(y_+) = \nu a_H y_+^m \leqslant \chi$ at $y_+ \leqslant y_{1+}$ which yields that $y_{1+} = (a_H Pr)^{-1/m}$. The next layer $y_{1+} \leqslant y_+ \leqslant y_{2+}$ is the "viscous sublayer" where once again $\varepsilon_H(y_+) = \nu a_H y_+^m$, but the molecular transfer of heat or matter can be neglected. Finally above the plane $y_+ = y_{2+}$ (with $a_H y_{2+}^m = y_{2+}/\alpha$) we find the "fully turbulent (logarithmic) layer" with the relations $\varepsilon_H(y_+) = \nu y_+/\alpha$ and $\Theta_+(y_+) = \alpha \ln y_+ + \beta(Pr)$. If we use the matching condition for $\Theta_+(y_+)$ at $y_+ = y_{1+}$ and $y_+ = y_{2+}$, we easily obtain that

$$\beta(Pr) = b_1 Pr^{(m-1)/m} + b_2 \tag{26}$$

in the suggested model, where b_1 and b_2 are numerical constants and

$$b_1 = \frac{m}{m-1} a_H^{-1/m}.$$

It is also possible to follow [73] and to use a slightly more accurate two-layer model in which both the molecular and the turbulent transfers are taken into account up to $y_+ = y_{2+}$, i.e. it is assumed that $\varepsilon_{ef} = \varepsilon_H + \chi = \nu a_H y_+^m + \chi$ at any $y_+ \leqslant y_{2+}$. If this model is used, an equation of form (26) will also be obtained, but in this case

$$b_1 = \left[1 - 2\sum_{k=1}^{\infty} \frac{(-1)^k}{m^2k^2 - 1}\right] a_H^{-1/m} \approx \frac{m^2+1}{m^2-1} a_H^{-1/m}.$$

The approximate interpolation equation for the Nusselt number for heat transfer in pipes was suggested in the book [13]. The equation has the same general form as equation (23) and can be interpreted as corresponding to an equation of the form

$$\beta(Pr) = b_1 Pr^{(m-1)/m} + \alpha \ln Pr + b_2. \tag{27}$$

The actual values of the parameters m, α, b_1 and b_2 used by Levich are inconsistent with the empirical data on the values of $\beta(Pr)$. (Such data were absent at the time when book [13] was written.) However, the very idea of the use of an interpolation formula of the form (27) which coincides with (26) at $Pr \gg 1$ and also has the correct asymptotic behavious at $Pr \ll 1$ seems quite reasonable.

The above discussion in this paper suggests that it is reasonable to set $m = 3$ and $\alpha = 2{\cdot}12$. The coefficient b_1 can be determined with a fair accuracy from the analysis of data of values of $\beta(Pr)$ at sufficiently high Pr which are collected in Table 1. There exists also quite a different way to determine the value of b_1. The equation $\varepsilon_M = \nu a_M y_+^3$ for eddy viscosity near a wall is clearly equivalent to the relation $-\overline{u'v'} = u_*^2 a_M y_+^3$ for the shear stress distribution near a wall. Accurate measurements of the shear stress distribution were made by Laufer [74] and Klebanoff [75]. The treatment of their data imply that apparently $-\overline{u'v'}/u_*^2 \approx 0{\cdot}001\, y_+^3$ in neighbourhood of a wall. The same result was deduced by Sleicher [44] from velocity and temperature profile measurements. If we accept that the turbulent Prandtl number is close to unity in the viscous sublayer (this is confirmed by the data of [44]) then we obtain that $a_H \approx 0{\cdot}001$. Hence it must be expected that

$$b_1 \approx \frac{10}{8} a_H^{-\frac{1}{3}} \approx 12{\cdot}5.$$

It is surprising that this seemingly quite rough estimate coincides exactly with that which is implied by experimental data of $\beta(Pr)$ from Table 1.

When the values of m, α and b_1 are determined, the value of b_2 can be found from the condition that $\beta(0{\cdot}7) = 3{\cdot}8$. This condition yields the result: $b_2 \approx -5{\cdot}3$. Hence we obtain the following final equation for $\beta(Pr)$ which will be used below

$$\beta(Pr) = 12{\cdot}5 Pr^{\frac{2}{3}} + 2{\cdot}12 \ln Pr - 5{\cdot}3. \tag{28}$$

Equation (28) is also plotted in Fig. 5. We see that it fits all the experimental data at $Pr \geqslant 0{\cdot}7$ quite satisfactorily.

When the form of the function $\beta(Pr)$ is known, we can use it for obtaining the general heat and mass transfer laws valid for an extensive range of Prandtl numbers. In particular, the generalization of law (25) for heat transfer through an air boundary layer for the case of arbitrary Pr has the following form

$$Nu = \frac{Re_x \, Pr\sqrt{(c_f/2)}}{2{\cdot}12 \ln (Re_x c_f) + 12{\cdot}5 \, Pr^{2/3} + 2{\cdot}12 \ln Pr - 7{\cdot}2}. \quad (29)$$

Equation (29) is plotted in Fig. 3 together with data of [29, 30] for heat transfer in water ($Pr = 3$ and $5{\cdot}3$) and technical oil ($Pr = 64$), the measurements in the boundary layer over a flat plate. We see that the agreement of the theory with the experiment is satisfactory. For a water boundary layer, the curves are also plotted in Fig. 3 which describe the empirical equation $Nu = 0{\cdot}0295 \, Pr^{0{\cdot}6} Re_x^{0{\cdot}8}$ suggested by Kays [15] for heat transfer calculations in the case of a boundary layer with $0{\cdot}5 \leqslant Pr \leqslant 10$.

The parameter values used in deriving equation (24) together with equation (28) yield the following heat and mass transfer law for pipe flow

$$Nu = \frac{Re \, Pr\sqrt{(C_f/2)}}{2{\cdot}12 \ln (Re\sqrt{C_f}) + 12{\cdot}5 \, Pr^{2/3} + 2{\cdot}12 \ln Pr - 10{\cdot}1}. \quad (30)$$

This equation is compared in Fig. 6 with numerous data on heat and mass transfer in smooth pipes taken from the references listed in Tables 2 and 5. (Figure 6 illustrates the scatter of the data points. To avoid overcrowding only a portion of the data points is plotted.) The data cover more than 6 orders of magnitude of Prandtl number variations and 2 orders of Reynolds number variations. In the cases when the experimental data in the original references were given for the values of Re which are intermediate between the values from the curves in Fig. 6, the data were interpolated with respect to Re between the nearest two available values. The scatter of the points in Fig. 6 is relatively great. This is especially so for heat transfer data at moderate values of Re as is expected (since one of the assumptions in this work requires high values of Re and some other assumptions are fulfilled more accurately for mass transfer than for heat transfer). However on the whole equation (30) fits satisfactorily the enormous collection of data used in Fig. 6.

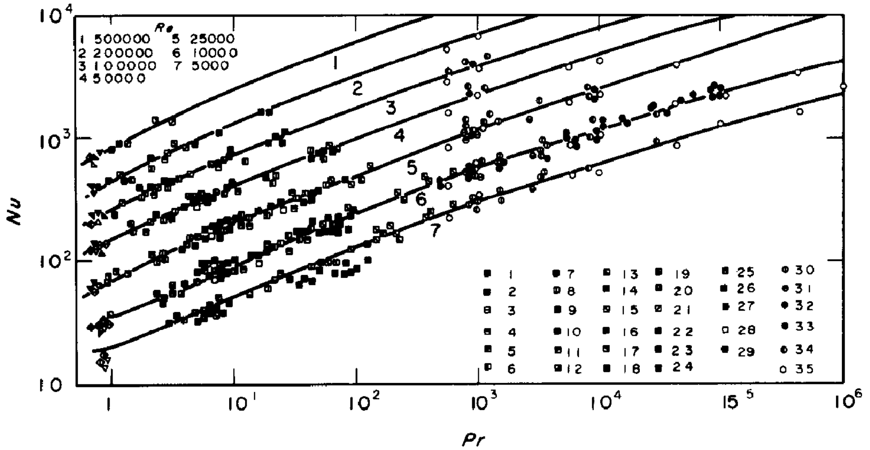

FIG. 6. Comparison of heat and mass transfer data for pipe flows with equation (30).

3.3 *Heat transfer at very low* Pr (*liquid metals*)

The data in Fig. 6 do not include measurements at $Pr \leqslant 0{\cdot}5$. If the Pr is very small (i.e. heat transfer in liquid metals) then equation (30) cannot be used, since it was derived by neglecting

Table 5. Heat and mass transfer studies for turbulent pipe flows

I. *Heat transfer*

No.	Authors and date	Ref.	Liquid	*Pr*	$Re \times 10^{-3}$
1	2	3	4	5	6
1.	Morris, Whitman (1928)	[76]	water: oil	3–701	5–45
2.	Eagle, Ferguson (1930)	[77]	water	3–10	10–100
3.	Sherwood, Petrie (1932)	[78]	water: oil	3·6–27	5–40
4.	Logan *et al.* (1934)	[79]	water: oil	2–50	7·5–250
5.	Cope (1937)	[80]	water	5·9–8	5–26
6.	Ullock, Badger (1937)	[81]	oil	14–370	5–90
7.	Norris, Sims (1942)	[82]	oil	35–140	5–11
8.	Bernardo, Eian (1945)	[83]	water: ethyleneglycol	1·4–60	5–200
9.	Lel'chuk (1950)	[84]	steam	1–1·1	100–1750
10.	Kaufmann, Iseley (1950)	[85]	water	2·8–12	10–50
11.	Grele, Gideon (1953)	[86]	molten sodium hydroxide	3·3–7·3	5–50
12.	Hoffman (1953)	[34]	molten sodium hydroxide	4·2–5·6	10
13.	Hartnett (1955)	[87]	water: oil	6·5–480	5–90
14.	Davies, Al-arabi (1955)	[88]	water	8·8	5–17
15.	Friend, Metzner (1958)	[34]	corn syrup: molasses	50–600	5–40
16.	Hastrup *et al.* (1958)	[89, 90]	water	1–8	50–250
17.	Alad'ev *et al.* (1959)	[91]	steam	0·9–1·4	10–25
18.	Mikheev (1959)	[46]	water	3·2–7	10–105
19.	Ivanovskii (1959)	[92]	water	4·1–7·2	10–43
20.	Ede (1961)	[48]	water	5·1–10·5	5–100
21.	Dipprey, Sabersky (1963)	[93]	water	1·2–59	14–500
22.	Malina, Sparrow (1964)	[94]	water	3–75	12–101
23.	Allen, Eckert (1964)	[95]	water	8	13–111
24.	Galin *et al.* (1965)	[96, 97]	water	3·4–8	5–180
25.	Hufschmidt *et al.* (1966)	[98]	water	2–5·5	20–640
26.	Shlykov, Leongardt (1966)	[99]	polyalkyl-benzol pitch	10–70	30–300
27.	Kalinin, Yarkho (1966)	[100, 101]	water: aqueous solutions of glycerin	3–50	5–100
28.	Gowen, Smith (1967)	[14]	ethylene-glycol	14·3	11–25
29.	Smith *et al.*	[31]	water	5·7	9–50

II. *Mass transfer*

No.	Authors and date	Ref.	Liquid and diffusing sublayer	*Pr*	$Re \times 10^{-3}$
1	2	3	4	5	6
30.	Linton, Sherwood (1950)	[102]	water: benzoic acid and cinnamic acid	960–3160	5–68
31.	Meyerink, Frienlender (1962)	[103]	water or aqueous solutions of sodium hydroxide: benzoic acid and cynnamic acid or aspirin	850–970	5–25
32.	Hamilton, Harriott (1965)	[66, 67]	water or aqueous solutions of glycerin: benzoic acid	930–100000	10–100
33.	Hanratty *et al.* (1963)	[104–107]	aqueous solution of sodium hydroxide: ferricyanide ion	2400	5–75
34.	Zarubin (1968)	[108]	$HCl + CaCl_2$: copper	560–28500	5–100
35.	Kader, Gukhman	[70, 71]	water or aqueous solutions of glycerin: benzoic acid	500–1000000	5–335

the contribution of the molecular transfer sublayer to the formation of the bulk temperature Θ_b. Moreover all of the liquid metal experiments were carried out with the condition of constant wall heat flux j_w, i.e. the boundary condition $\Theta(0) = \Theta_w =$ constant has to be replaced here by the condition $-c_p\rho\, \mathrm{d}\Theta(0)/\mathrm{d}y = j_w =$ constant. It is known that if $Pr > 0{\cdot}5$ the values of the quantities c_h and Nu are practically the same at $\Theta_w =$ constant and at $j_w =$ constant (see [15, 109]). Hence all of the above results can be applied to both cases with equal validity. (In fact some of the experimental points in Fig. 6 were obtained from the measurements at $j_w =$ constant, and not at $\Theta_w =$ constant.) However if $Pr \ll 1$ then the values of c_h and Nu differ significantly in the cases when $\Theta_w =$ constant and when $j_w =$ constant (see e.g. [15]). Moreover we cannot also exclude the possibility that the term b_2 in equation (27) is in fact a slowly varying function of Pr which variation can be neglected only in the range $Pr > 0{\cdot}5$ where it cannot evidently play a role. Therefore it is not surprising at all that by applying equation (27) to data on heat transfer in liquid metals (i.e. to $Pr \ll 1$) the best fit is attained when the value of the constant term b_2 is changed and equation (28) is replaced by the equation

$$\beta(Pr) = 12{\cdot}5\, Pr^{2/3} + 2{\cdot}12 \ln Pr - 1{\cdot}5. \qquad (31)$$

The constant term in equation (31) was determined by comparison of the calculated values of $\beta(Pr)$ with the experimental data in Table 1 for temperature profiles in mercury flows (see the left side of the Fig. 3 where the continuous line corresponds to equation (31)).

Equation (31) permits one to calculate the coefficient c_h quite easily with the aid of the general equation (16). However, we cannot now use the simple equation (20) for the correction factor $(\Theta_w - \Theta_b)/(\Theta_w - \Theta_1)$ when transforming from c_h to C_h. Instead we must use a more complicated model of the temperature profile which takes into account the existence of a molecular diffusivity sublayer (which is clearly thick when $Pr \ll 1$, i.e. χ is very high). We can use the simplest two-layer model of Taylor and Prandtl as a first approximation. According to

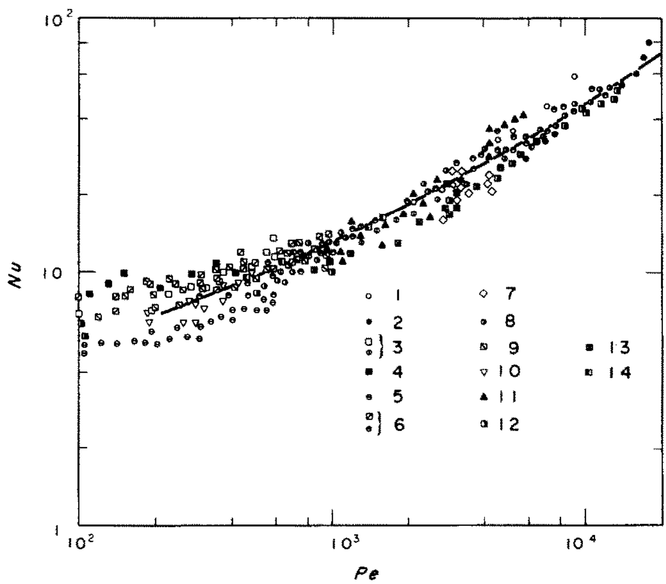

FIG. 7. Comparison of heat transfer data in liquid metals with equation (33). All the numbers correspond to references in Table 6.

this model $\Theta_+(y_+) = Pr\, y_+$ for $y_+ \leqslant a_1/Pr$ where a_1 is the dimensionless constant of the same order as the dimensionless thickness of the viscous sublayer (we have set $a_1 = 5$ in our calculations), and $\Theta_+(y_+) = \alpha \ln y_+ + \beta(Pr)$ for $y_+ > a_1/Pr$. By use of such a form of the temperature profile near to the wall and also assuming that the logarithmic form of the temperature defect law (with $\beta_1 = 0$) is valid for $y > a_1\chi/u_*$, we find that the following equation is valid with a sufficient degree of accuracy:

$$\frac{1}{\Delta} = \frac{\Theta_w - \Theta_b}{\Theta_w - \Theta_1} = (1 - \eta_1)^2 + \sqrt{(C_f/2)}]$$

$$- \frac{\Theta_*}{\Theta_w - \Theta_1}[3{\cdot}2(1 - 2\eta_1) - 7\sqrt{(C_f/2)}] \qquad (32)$$

where $\eta_1 = 10\sqrt{(2)}/Pe\sqrt{(C_f)}$. (If the right-hand side of (32) is smaller than 0·5 which corresponds to a laminar pipe flow then it is, of course, reasonable to replace Δ^{-1} by 0·5.) It follows from equations (16) and (31) that $Nu = C_h Pe$ can be calculated from the equation

$$Nu = \frac{Re\, Pr\sqrt{(C_f/2)}\Delta}{2{\cdot}12 \ln (Re\sqrt{(C_f)}) + 12{\cdot}5Pr^{\frac{3}{4}} + 2{\cdot}12 \ln Pr + 3{\cdot}1}. \qquad (33)$$

where the correction factor Δ is determined by the equation (32). This equation is compared in Fig. 7 with data from experimental works on heat transfer in liquid metal pipe flows listed in Table 6. The data in Fig. 7 fill the range of Prandtl numbers from 6×10^{-3} to 3×10^{-2} and since the calculations show that Nu in this range depends practically on $Pe = RePr$ only and not on Re and Pr separately, we can use a single theoretical curve for the entire range of Pr (the continuous line in Fig. 7). The heat transfer data for liquid metals are very scattered and poorly reproducible: therefore in Fig. 7 only data were used of experiments which either were accompanied by the measurement of the mean temperature distribution (which yields additional control of the data) or were quite recent and seemed to be accurate. We see that the calculations with the aid of equation (33) fit satisfactorily all of the selected data and are also close to the empirical calculation procedure recommended by Lyon [122] which is considered at present by most of the experts in the field as the best procedure for calculation of heat transfer parameters in liquid metals. Of course, we must bear in mind that the accuracy of all of the data on heat transfer in liquid

Table 6. Heat transfer studies for turbulent flows of liquid metals in pipes

No.	Authors and date	Ref.	Metal	Pr	$Re \times 10^{-3}$	Pe
1.	Isakoff, Drew (1951)	[110]	mercury	0·025	80–400	2000–9000
2.	Brown *et al.* (1957)	[111]	mercury	0·025	250–800	5800–18000
3.	Kirillov *et al.* (1959)	[112–113]	sodium, potassium,	0·015	7–129	50–1600
			mercury	0·026	25–270	600–7000
4.	Pirogov (1960)	[114]	sodium	0·006	23–70	17–416
5.	Petukhov, Yushin (1961)	[115]	mercury	0·024	0·6–23	14–600
6.	Subbotin *et al.* (1961)	[116]	sodium, potassium,	—	—	85–1000
			mercury	—	—	230–13000
7.	Kokorev, Ryaposov (1962)	[19–20]	mercury	0·026	107–167	2800–4300
8.	Subbotin *et al.* (1963)	[21]	mercury	0·026	22–427	540–12100
9.	Borishanskii *et al.* (1963)	[117]	sodium	0·006	16–128	120–940
10.	Filimonov *et al.* (1964)	[118]	aluminium	0·014	12–35	172–485
11.	Borishanskii *et al.* (1964)	[18]	mercury	0·022	47–262	1160–5600
12.	Skupinski *et al.* (1965)	[119]	sodium–potassium	0·015	3·6–950	58–13100
13.	Talanov, Ushakov (1967)	[120]	sodium–potassium	0·025	28–320	850–6400
14.	Hochreiter, Sesonske (1969)	[121]	sodium–potassium	0·025	18–65	480–1460

metals is quite low in comparison, say, with the accuracy of the data on mass transfer in ordinary liquids; therefore the coefficients of equation (33) cannot be considered as determined with a high degree of precision.

Let us stress finally that the above method of the determination of heat and mass transfer laws can be also applied to many more complicated situations. It should be remembered for example, that it was shown by Millikan [3] and Mises [4] that a similar derivation of the skin friction law can be easily established also for fully turbulent flow along a rough wall and for turbulent flow in a pipe of non-circular cross section. The same is clearly true for the derivation of heat and mass transfer law. Related arguments can be used also for a boundary layer in the presence of strong adverse pressure gradient and for heat transfer problems where buoyancy plays an important role. All of these questions need however a special investigation and we shall not linger on them here.

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LOIS DE TRANSFERT THERMIQUE ET MASSIQUE POUR DES ECOULEMENTS PARIETAUX ENTIEREMENT TURBULENTS

Résumé—La méthode générale d'Izakson et Millikan pour l'obtention de la loi classique de Prandtl–Nikuradse pour le frottement superficiel est appliquée à l'analyse du transfert thermique et massique turbulent dans des tubes, des canaux et des couches limites. La formule donnant le coefficient (ou le nombre de Nusselt) de transfert thermique (ou massique) contient, en tant que paramètres, les coefficients sans dimension des équations logarithmiques universelles des profils de vitesse et de température. Un de ces paramètres est une fonction universelle du nombre de Prandtl (ou de Schmidt) et tous les autres sont des constantes.

Les profils de vitesse et de température expérimentaux existants dans différents écoulements pariétaux turbulents permettent avec une bonne précision la détermination de tous les coefficients nécessaires. Les calculs qui en résultent sont en accord satisfaisant avec les nombreuses expériences sur le transfert thermique et massique dans des tubes et des couches limites sur plaque plane, pour un domaine de nombre de Prandtl (ou de Schmidt) compris en $6 . 10^{-3}$ et 10^{6} et pour une variation du nombre de Reynolds (ou Péclét) couvrant deux ordres de grandeur.

GESETZE DER WÄRME- UND STOFFÜBERTRAGUNG FÜR VOLLTURBULENTE WANDSTRÖMUNGEN

Zusammenfassung—Die allgemeine Methode von Izakson und Millikan zur Ableitung des bekannten Prandtl–Nikuradse-Reibungsgesetzes wird auf die Analyse der turbulenten Wärme- und Stoffübertragung in Rohren, Kanälen und Grenzschichten angewandt. Die abgeleitete Formulierung des Wärme (Stoff)-übergangskoeffizienten (oder der Nusselt-Zahl) enthält als Parameter die dimensionslosen Koeffizienten der universellen logarithmischen Gleichung für die Geschwindigkeits- und Temperaturprofile. Einer dieser Parameter ist eine universelle Funktion der Prandtl-(oder Schmidt)Zahl und alle anderen sind Konstanten. Die vorhandenen Geschwindigkeits- und Temperatur-Profil-Messungen in verschiedenen turbulenten Wandströmungen erlauben die Festlegung aller notwendigen Koeffizienten mit guter Genauigkeit. Die resultierenden Ergebnisse sind in zufriedenstellender Übereinstimmung mit zahlreichen experimentellen Untersuchungen über den Wärme- und Stofftransport in Rohren und an Grenzschichten an einer ebenen Platte für Prandtl-(oder Schmidt)Zahlen im Bereich von 6×10^{-3} bis 10^{6} und über zwei Grössenordnungen der Reynolds-(oder Péclét)Zahl.

ЗАКОНЫ ТЕПЛО-И МАССООБМЕНА ДЛЯ ТУРБУЛЕНТНЫХ ТЕЧЕНИЙ ВДОЛЬ СТЕНКИ

Аннотация—Общий метод, развитый Изаксоном и Милликеном для вывода закона поверхностного трения для турбулентных течений в трубах и каналах, применяется для получения закона тепло- и массопереноса при турбулентных течениях в трубах, каналах и пограничных слоях. Найденная формула для коэффициента теплоперецачи (или числа Нуссельта) содержит в качестве параметров безразмерные коэффициенты универсальных логарифмических формул для профилей скорости и температуры, один из которых является универсальной функцией числа Прандтля (или Шмидта), а остальные являются константами. Имеющиеся в настоящее время данные измерений профилей скорости и темп ратуры в турбулентных течениях вдоль стенки позволяют определить все нужные коэффициенты с приличной точностью. Получаемые при этом общие законы тепло- и массопереноса в трубе и пограничном слое на пластинке удовлетворительно согласуются с многочисленными экспериментами в пределах диапазона чисел Прандтля от $6 \cdot 10^{-3}$ до 10^{6} и на протяжении двух порядков изменения числа Рейнольдса (или Пекле).